Robust optimization of distributed parameter systems

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Abstract

In this paper, we will discuss the use of new methods from robust control and especially H^∞ theory for the explicit construction optimal feedback compensators for several practical distributed parameter systems.

Indeed, based on operator and interpolation theoretic methods one can now solve the standard H^∞ control problem for a broad class of systems modelled by PDEs. In our approach, the complexity of the computations involved is only a function of the weighting filters, and not the state space dimension which is why we can handle infinite dimensional systems with no approximations involved. These techniques are based on certain operator theoretic notions connected with a class of operators which we call *skew Toeplitz.* These are precisely the operators which appear in the H^∞ optimization problem.

Keywords: Robust control, H^∞ optimization, distributed parameter systems, skew Toeplitz operators.

1 INTRODUCTION

This paper is concerned with a survey on the the utilization of certain methods from generalized interpolation and functional analysis to attack a wide range of problems in robust control. In all our analyses and designs, we will work in the frequency domain. The paper will have a tutorial nature in order to explicate these techniques. Thus we will not give the most general results which can be derived using these methods, but will content ourselves to giving several interesting applications. The most general theory may be found in the appropriate references which will be given below.

In order to fix ideas, let us consider now the *robust stabilization problem* for systems with parameter uncertainty. In this framework, we are given a continuously parametrized family P_k of plants where the parameter vector k varies in some fixed compact parameter set K. Then we want to design a fixed feedback controller C such that for each $k \in K$, the standard closed feedback configuration will be internally asymptotically stable.

This problem is very hard [28], and no general solution is known even for the case in which the P_k are restricted to be finite dimensional, linear time-invariant (LTI), single input/single

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output (SISO) systems. However, for special cases of importance in practical design involving parameter uncertainty modelled as certain multiplicative or additive perturbations of a nominal plant, one can give a complete algorithmic solution.

Moreover, it turns out that this class of robust stabilization problems can be equivalently formulated as a certain weighted sensitivity H^∞-optimization problem when the plants are finite dimensional, LTI, SISO, and the internally stabilizing controllers are taken to be also in this class. Indeed, in the weighted sensitivity optimization problem, for given plant $P(s)$, and stable filter ("weight") W(s), one is required to minimize the H^∞-norm of the weighted sensitivity function $W(s)(1 + P(s)C(s))^{-1}$ over the set of causal stabilizing feedback compensators C(s). The point is however, that the kind of robust stabilization problem we have just mentioned and weighted sensitivity minimization both amount to classical interpolation problems of the Nevanlinna-Pick type.

For multivariable and distributed systems the situation is much more complex. Indeed, even though the techniques used to solve the unweighted sensitivity and gain optimization problems for SISO, LTI finite dimensional systems go through quite easily for a large class of distributed systems already for a first order weight and a plant just consisting of a pure delay, the problem of weighted sensitivity H^∞-minimization seems to genuinely reflect the infinite-dimensional nature of the plant [16] as we will see below. For such problems, we have developed a general operator based methodology which we call *skew Toeplitz theory.* This will be sketched below.

Almost all the work described in this paper is joint. Among my many collaborators given in alphabetical order are: Hari Bercovici, Ciprian Foias, Bruce Francis, Art Frazho, Pramod Khargonekar, Tanya Lypchuk, Hitay Özbay, M. Clive Smith, and George Zames. Finally, there is a rathere comprehensive treatment of the material discussed below in the forthcoming monograph [12].

2 SKEW TOEPLITZ THEORY

In this section, we will sketch a frequency domain (*skew Toeplitz*) approach in the H^∞ optimization of distributed systems. This approach leads to an explicit solution of the standard (four block) problem for a broad class of distributed multivariable systems [25], [23]. We shall however concentrate in this tutorial paper on solving the two block (mixed sensitivity) problem for unstable distributed SISO plants [24], and the four block problem for stable SISO distributed plants, or arbitrary lumped SISO systems [13], [14]. We shall also emphasize the computational aspects of this methodology which allows one to reduce infinite dimensional design problems to finite dimensional matrix and polynomial operations. There are two nice MatLab packages available to implement this theory: one written by Kathryn Lenz and Handong Tu, and the other by Hitay Özbay and O. Toker.

The essence of the skew Toeplitz methodolgy is the Sz. Nagy-Foias Commutant Lifting Theorem [27], which allows one to reduce an H^∞-optimization problem to the computation of the norm of a certain *skew Toeplitz* operator. The Sarason theorem [26] (which we will employ below) is special case of this result. Because of the technical nature of the Commutant Lifting

Theorem, we shall not state it here in full generality but simply content ourselves with the special Sarason formulation. However, based on this theorem and the resulting skew Toeplitz methodology (see [1]), [18]), one may reduce the computation of the optimal performance and controller of very general (distributed) systems to the computation of the singular values of a **finite** Hermitian matrix, the size of which depends on the MacMillan degree of the weighting filters (taken to be finite dimensional, of course), and the number of unstable poles of the plant (which we assume to be finite). Hence we reduce an infinite dimensional optimization problem to a finite dimensional computation. These methods have already been applied to delay systems [17], [9], as well as flexible beam problems (modelled by the Euler-Bernoulli equation with Kelvin-Voigt damping) [22].

2.1 Brief review of skew Toeplitz theory

In this section, we will give the relevant aspects of the skew Toeplitz computational method as applied to the two block problem for unstable distributed plants. In fact, we will show that several two block H^∞-minimization problems reduce to the computation of the norm of a certain skew Toeplitz operator. For applications of skew Toeplitz theory to other types of H^∞ problems (one block and four block); see [1], [13], [14], [18], and [24]. For a general discussion on the two block problem see [8]. The discussion given here is based on [24].

We begin with some notation. The Hardy spaces H^2 and H^∞ are defined on the unit disc in the standard way. We denote

$$\begin{aligned} \tilde{H}^\infty &:= \left\{ f \in H^\infty : \overline{f(\overline{z})} = f(z) \right\} \\ R\tilde{H}^\infty &:= \left\{ \text{rational functions in } \tilde{H}^\infty \right\} \end{aligned}$$

We consider the feedback configuration of Figure 1 with

$$P = \frac{G_n}{G_d}$$

and $G_n \in \tilde{H}^\infty$, $G_d \in R\tilde{H}^\infty$.

We assume that (i) $G_n = m_n G_{no}$, where $m_n \in \tilde{H}^\infty$ is inner (arbitrary) and $G_{no} \in \tilde{H}^\infty$ is outer, and (ii) G_n, G_d have no common zeros in the closed unit disc. We also write $G_d = m_d G_{do}$ where $m_d \in R\tilde{H}^\infty$ is inner and $G_{do} \in R\tilde{H}^\infty$ is outer. Under these assumptions there exist $X \in R\tilde{H}^\infty$ and $Y \in \tilde{H}^\infty$ such that

$$X G_n + Y G_d = 1. \tag{1}$$

The set of all controllers which stabilize the plant can now be written in the form

$$C = \frac{X + Q G_d}{Y - Q G_n}$$

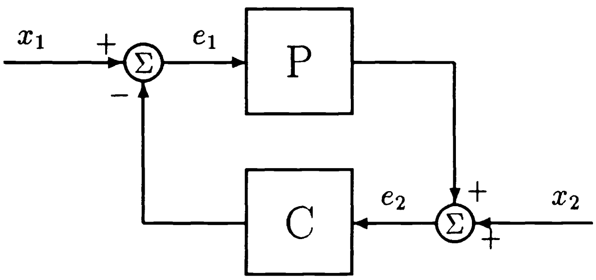

Figure 1: Standard Feedback Configuration

for some $Q \in \tilde{H}^\infty$. Now let $S := (1 + PC)^{-1}$ and note that

$$S = 1 - XG_n - QG_nG_d. \tag{2}$$

In [24], we show that the computation of

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2(S-1) \end{bmatrix} \right\|$$

where $W_1, W_2 \in R\tilde{H}^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in R\tilde{H}^\infty$ may be reduced to computing the norm of the "skew Toeplitz" operator (this follows from the Commutant Lifting Theorem [24]),

$$\mathbf{A} := \begin{bmatrix} \mathbf{P}_{H(m_v)} \left(W_0(\mathbf{S}) - \hat{W}_0(\mathbf{S}) m(\mathbf{S}) \right) \\ G_0(\mathbf{S}) \end{bmatrix}, \tag{3}$$

where $\mathbf{S} : H^2 \to H^2$ denotes the unilateral shift, $H(m_v) := H^2 \ominus m_v H^2$ and $\mathbf{P}_{H(m_v)}$ the orthogonal projection onto $H(m_v)$, for m, m_v inner functions associated to the plant and weighting filters, and where $W_0, \hat{W}_0, G_0$ are rational H^∞ functions computed from the plant and weighting filters. (In this section and the next, we use the bold $\mathbf{S}$ to denote the unilateral shift in order to avoid confusion with S which will stand for sensitivity.) This reduction is true for plants with *arbitrary* outer parts. We can do a similar type of reduction for the following 2-block minimization problem in case the outer part of the numerator of the plant is rational. (See [24] for details.) Find

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2 CS \end{bmatrix} \right\| \tag{4}$$

where $W_1, W_2 \in R\tilde{H}^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in R\tilde{H}^\infty$.

In [24], the following result is proven:

Theorem 1 *Let n denote the maximum of the MacMillan degrees of the weighting filters W_1 and W_2, and let ℓ denote the number of unstable poles of the plant P. Then the singular values of $\mathbf{A}$ may be derived from an explicitly computable system of $3n + 2\ell$ linear equations (the "singular system.") Moreover, from this system the corresponding singular vectors may also be found.*

The singular system of equations is written down in [24]. It is based on the previous singular system derived in [13]. The computation of the *maximal* singular value and the associated singular vectors of $\mathbf{A}$, then allows us to find the optimal performance μ of our original control problem and the corresponding optimal compensator.

3 STANDARD PROBLEM

We will complete this paper with an outline of how to solve the most general H^∞ synthesis problem using the above generalized interpolation ideas. Indeed, we will show how these methods may be used to solve a very general case of the **standard** or **four block problem** in H^∞ design valid for a large class of distributed systems.

The motivations for studying the H^∞ optimization in systems theory lie in the most natural problems of control engineering such as robust stabilization, sensitivity minimization, and model matching. It can be shown that, in the sense of H^∞ optimality, these problems are equivalent, and can be formulated as one *standard problem* [19]. More precisely, consider the feedback diagram in Figure 2. In this configuration w, u, y, and z are vector-valued signals with w the exogenous input representing the disturbances, measurement noises etc., u the command signal, z the output to be controlled, and y the measured output. G represents a combination of the plant and the weights in the control system. The standard H^∞ problem is to find a stabilizing controller K such that the H^∞ norm of the transfer function from w to z is minimized.

Now it is quite well-known that an optimal solution of the standard problem can be reduced to finding the singular values and vectors of a certain operator (the so-called **four block operator**) which will be defined below. Depending on the specific problem considered, the corresponding four block operator can be simplified to a 2-block or a 1-block operator.

Besides appearing in the most general H^∞ synthesis problems, the four block operators also have a number of intriguing mathematical properties in the sense that they are natural extensions of both the Hankel and Toeplitz operators. For this reason they fit into the skew Toeplitz framework. For the full details of our arguments and details about the skew Toeplitz theory applied to this problem (for multivariable systems) we refer the reader to [1], [23], [25]. Here we will just consider the four block problem for single input/single output systems as in [13] and [14].

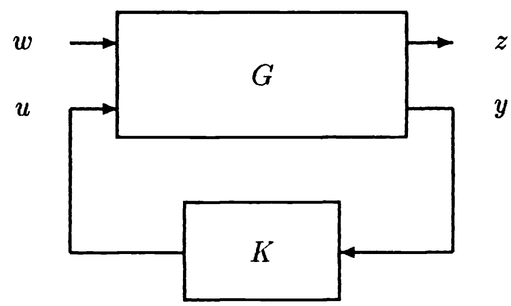

Figure 2

More precisely, invoking the Youla parametrization and employing standard manipulations involving inner-outer factorizations, for a large class of distributed systems we may reduce the standard problem mentioned above to the following mathematical one. Let $w, f, g, h, \in H^\infty$, where w, f, g, h are rational and m is nonconstant inner. (All of our Hardy spaces will be defined on the unit disc D in the standard way.) Set

$$\mu := \inf\{\| \begin{bmatrix} w - mq & f \\ g & h \end{bmatrix} \|_\infty : q \in H^\infty\}. \tag{5}$$

Then we want to give an algorithm for calculating the quantity μ, and for finding the corresponding optimal $q_{opt} \in H^\infty$, i.e., q_{opt} is such that

$$\mu := \| \begin{bmatrix} w - mq_{opt} & f \\ g & h \end{bmatrix} \|_\infty .$$

Note that for $f = g = h = 0$, this reduces to the classical Nehari problem.

As mentioned above, we can identify μ as the norm of a certain "four block operator" (see Section 5.1 for the precise definition), and then in Sections 5.2 and 5.3 give a determinantal formula for its computation.

3.1 The four block operator

We will now define the *four block operator* which will be the major mathematical object of study in the rest of this paper. Let $H(m) := H^2 \ominus mH^2$, $L(m) := L^2 \ominus mH^2$, and we let $P_{H(m)} : H^2 \to H(m)$, $P_{L(m)} : L^2 \to L(m)$ denote the corresponding orthogonal projections. Let $S : H^2 \to H^2$ denote unilateral shift, $T : H(m) \to H(m)$ the compression of S, and let $U : L^2 \to L^2$ denote bilateral shift, with $T(m) : L(m) \to L(m)$ the compression of U. Then for $w, f, g, h \in H^\infty$ rational, we set

$$A := \begin{bmatrix} P_{L(m)} w(S) & P_{L(m)} f(U) \\ g(S) & h(U) \end{bmatrix} .$$

Note that

$$A = \begin{bmatrix} w(T)P_{H(m)} & f(T(m))P_L(m) \\ g(S) & h(U) \end{bmatrix}$$

(Clearly $A : H^2 \oplus L^2 \to L(m) \oplus L^2$.) The Commutant Lifting Theorem then allows us to identify

$$\|A\| = \mu.$$

Thus in order to solve the four block problem we are required to compute the norm of the operator A. This we will show how to do in the next two sections.

In order carry out this program, we will first need to identify the essential norm of A (denoted by $\|A\|_e$). We are using the standard notation from operator theory as, for example, given in [27]. In particular σ_e will denote the essential spectrum, and $A(\overline{D})$ will stand for the set of analytic functions on D which are continuous on the closed disc $\overline{D}$. We can now state the following result whose proof we refer the reader to [14]:

Theorem 2 *Notation as above. Let $w, f, g, h \in A(\overline{D})$, and set*

$$\alpha := \max\{\| \begin{bmatrix} w(\zeta) & f(\zeta) \\ g(\zeta) & h(\zeta) \end{bmatrix} \| : \zeta \in \sigma_e(T)\} \tag{6}$$

$$\beta := \max\{\| \begin{bmatrix} 0 & 0 \\ g^{(\zeta)} & h(\zeta) \end{bmatrix} \| : \zeta \in \partial D\} \tag{7}$$

$$\gamma := \sup\{\| \begin{bmatrix} f(\zeta) \\ h(\zeta) \end{bmatrix} \| : \zeta \in \partial D\}. \tag{8}$$

Then

$$\|A\|_e = \max(\alpha, \beta, \gamma). \tag{9}$$

3.2 Singular system

In this section, we will study the invertibility of certain skew Toeplitz operators as considered in [14] which occur as basic elements in our procedure for computing the norm and singular values of the four block operator. We will show that the calculation of the singular values of the four block operator A amounts to inverting two ordinary Toeplitz operators, and essentially inverting an associated skew Toeplitz operator. The Fredholm conditions on the invertibility of the skew Toeplitz operator (which is essentially invertible), and the coupling between the various systems (expressed as **matching conditions**) constitutes a certain linear system of equations called the **singular system** which allows one to determine the invertibility of A.

Using the notation of Section 5.1, we let $\rho > \max(\alpha, \beta, \gamma)$. Note that when $\|A\| > \|A\|_e$, $\|A\|^2$ is an eigenvalue of AA^*. By slight abuse of notation, ζ will denote a complex variable as well as an element of ∂D (the unit circle). The context will always make the meaning clear. Of course, if $\zeta \in \partial D$, then $\overline{\zeta} = 1/\zeta$.

As above, we take w, f, g, h to be rational, and so we can express $w = a/q$, $f = b/q$, $g = c/q$, $h = d/q$, where a, b, c, d, q are polynomials of degree $\leq n$. Then we have that

$$A := \begin{bmatrix} P_{L(m)}(\frac{a}{q})(S) & P_{L(m)}(\frac{b}{q})(U) \\ (\frac{c}{q})(S) & (\frac{d}{q})(U) \end{bmatrix}.$$

Now ρ^2 is an eigenvalue of AA^* if and only if

$$\begin{bmatrix} \rho^2 q(T(m))q(T(m))^* & 0 \\ 0 & \rho^2 q(U)q(U)^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \tag{10}$$

$$\begin{bmatrix} P_{L(m)}a(S) & P_{L(m)}b(U) \\ c(S) & d(U) \end{bmatrix} \begin{bmatrix} a(S)^*P & Pc(U)^* \\ b(U)^*P_L(m) & d(U)^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

for some non-zero

$$\begin{bmatrix} u \\ v \end{bmatrix} \in L(m) \oplus L^2$$

where $P : L^2 \to H^2$ denotes orthogonal projection.

Set

$$u_+ := Pu, \;\; u_- := (I - P)u$$

and

$$v_+ := Pv, \;\; v_- := (I - P)v, \;\; v_{++} := (I - P_{H(m)})v.$$

Then we can write (10) equivalently as

$$\mathcal{C} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} a(T)a(T)^* & a(T)P_{H(m)}c(S)^* \\ c(S)a(T)^* & c(S)c(S)^* \end{bmatrix} \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} = 0 \tag{11}$$

where

$$\mathcal{C} :=$$

$$\begin{bmatrix} \rho^2 q(T(m))q(T(m))^* - b(T(m))b(T(m))^* & -b(T(m))P_{L(m)}d(U)^* \\ -d(U)b(T(m))^* & \rho^2 q(U)q(U)^* - d(U)d(U)^* \end{bmatrix}.$$

Let $V := U^*|L^2 \ominus H^2$. If we apply $(I - P)$ to both rows of (11), we see that the basic block operator applied to

$$\begin{bmatrix} u_- \\ v_- \end{bmatrix}$$

is

$$C_- \begin{bmatrix} u_- \\ v_- \end{bmatrix} := \tag{12}$$

$$\begin{bmatrix} \rho^2 q(V^*)q_*(V) - b(V^*)b_*(V) & -b(V^*)d_*(V) \\ -d(V^*)b_*(V) & \rho^2 q(V^*)q_*(V) - d(V^*)d_*(V) \end{bmatrix}.$$

$$\begin{bmatrix} u_- \\ v_- \end{bmatrix}$$

Next applying $(I - P_{H(m)})$ to both rows of (11), we see that the basic operator applied to v_{++} is

$$C_{++}\overline{m}v_{++} := P\{(\rho^2|q|^2 - |c|^2 - |d|^2)\overline{m}v_{++}\}. \tag{13}$$

Finally, applying $P_{H(m)}$ to (11), we derive that the basic operator applied to

$$\begin{bmatrix} u_+ \\ P_{H(m)}v_+ \end{bmatrix}$$

is

$$C_+ = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where

$$\begin{aligned}
c_{11} &:= \rho^2 q(T)q(T)^* - b(T)b(T)^* - a(T)a(T)^* \\
c_{12} &:= -b(T)P_{H(m)}d(S)^* \\
c_{21} &:= -d(T)b(T)^* \\
c_{22} &:= \rho^2 q(T)q(T)^* - d(T)d(T)^* - c(T)c(T)^*
\end{aligned}$$

The operators C_-, C_{++}, C_+ are all skew Toeplitz. In [14] we show how to invert C_- and C_{++} under the assumption $\rho > \|A\|_e$. The essential inversion of C_+ can be handled exactly as in [14], [18]. The resulting finite system of linear equations (the "singular system") then leads to the following result:

Theorem 3 *There exists an explicitly computable* $5n \times 5n$ *Hermitian matrix* $M(\rho)$ *such that* $\overline{\rho} > \max\{\alpha, \beta, \gamma\}$ *is a singular value of the four block operator* A *if and only if*

$$\det M(\overline{\rho}) = 0.$$

3.3 On optimal compensators

The above procedure also gives a way of computing the optimal compensator in a given four block problem . Indeed, from the above determinantal formula one can compute the Schmidt pair ψ, η corresponding to the singular value $s := \|A\|$ when $s > \|A\|_e$. We will indicate how one derives the optimal interpolant (and thus the opimal compensator) from these Schmidt vectors. (For more details see [15].) In order to do this, notice

$$A\psi = s\eta.$$

Thus, there exists $q_{opt} \in H^\infty$ with

$$(w - q_{opt})\psi_1 + f\psi_2 = s\eta_1$$

$$g\psi_1 + h\psi_2 = s\eta_2$$

where

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

One can show (see [15]), that $\psi_1 \neq 0$, so that

$$q_{opt} = w - \frac{s\eta_1 - f\psi_2}{\psi_1}.$$

Note from q_{opt}, using the Youla parametrization, we can derive the corresponding optimal controller in a given systems design problem. See also [15] for an extension of the theory of Adamjan-Arov-Krein (valid for the Hankel operator) to the singular values of the four block operator and their relationship to more general interpolation and distance problems.

4 CONCLUDING REMARKS

In this paper, we have outlined several key problems from robust control that may be successfully solved using frequency domain methods based on generalized interpolation theory and the Commutant Lifting Theorem. We would like to conclude this paper with some remarks about a completely different direction in interpolation theory again motivated by control problems.

The basic underlying structure in all that we have described above is interpolation on the disc by *norm bounded* analytic (possibly operator-valued) functions. This is the fundamental problem treated by classical commutant lifting theory, and is precisely the problem which arises in H^∞ control. Now while in the SISO case stability margin optimization and H^∞ synthesis reduce to the same interpolation theoretic problem (see Section 2 above), in the matrix case, this is false (see [29]). In fact, for multivariable systems, stability margin optimization leads to a novel interpolation problem, in which one does not look to bound the norm but instead the *spectral radius* of the interpolants [29]. These ideas of spectral interpolation and a spectral commutant lifting theorem are worked out in [2] and [3]. Finally, there has even been a extension of commutant lfting theory to the structured singular value of Doyle-Safonov which may lead to an analytic procedure for performing μ-synthesis. This is discussed in [4, 5, 6].

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